Exercise 7.8.4

ODEs of the form $y = xy' + f(y')$ are known as Clairaut equations. The first step in solving an equation of this type is to differentiate it, yielding

$$y' = y' + xy'' + f'(y')y'', \quad \text{or} \quad y''\left(x + f'(y')\right) = 0.$$

Solutions may therefore be obtained both from $y'' = 0$ and from $f'(y') = -x$. The so-called general solution comes from $y'' = 0$. For $f(y') = (y')^2$,

(a) Obtain the general solution (note that it contains a single constant).

(b) Obtain the so-called singular solution from $f'(y') = -x$. By substituting back into the original ODE show that this singular solution contains no adjustable constants.

Note. The singular solution is the envelope of the general solutions.

Solution

Suppose that $f(y') = (y')^2$. Then the general Clairaut equation becomes

$$y = xy' + (y')^2. \tag{1}$$

Differentiate both sides with respect to x.

$$y' = y' + xy'' + 2(y')y''$$

Subtract both sides by y'.

$$0 = xy'' + 2y'y''$$

Factor y''.

$$0 = y''(x + 2y')$$

By the zero product theorem,

$$\begin{aligned} y'' &= 0 & &\text{or} & x + 2y' &= 0 \\ y' &= A & &\text{or} & y' &= -\frac{x}{2} \\ y(x) &= Ax + B & &\text{or} & y(x) &= -\frac{x^2}{4} + C. \end{aligned}$$

Substitute these solutions back into equation (1) to determine the constants.

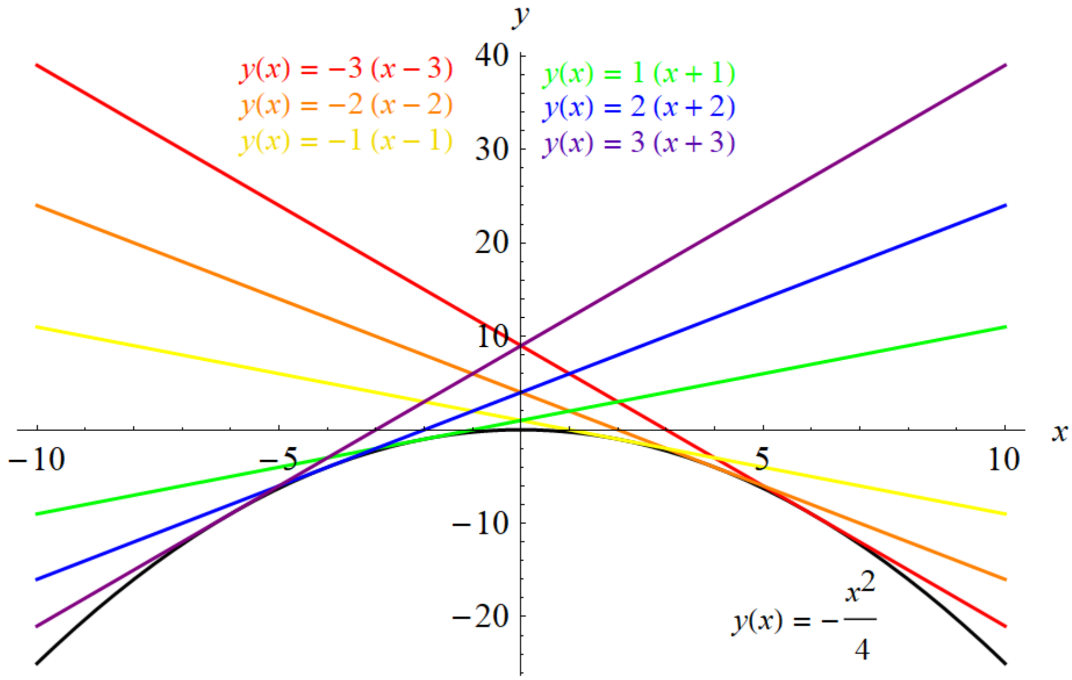
$$\begin{aligned} (Ax + B) &= x(A) + (A)^2 & \left(-\frac{x^2}{4} + C\right) &= x\left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 \\ B &= A^2 & C &= 0 \end{aligned}$$

Therefore, the general and singular solutions are, respectively,

$$y(x) = A(x + A) \quad \text{and} \quad y(x) = -\frac{x^2}{4}.$$

The singular solution contains no adjustable constant.

Below is a plot of the general solution for several values of A and the singular solution.



We see that the value of A determines where the general solution is tangent to the singular solution. The latter therefore acts as an envelope.